

On the Sign-imbalance of Permutation Tableaux

Joanna N. Chen¹, Robin D.P. Zhou²

¹College of Science
Tianjin University of Technology
Tianjin 300384, P.R. China

²College of Mathematics Physics and Information
Shaoxing University
Shaoxing 312000, P.R. China

¹joannachen@tjut.edu.cn, ²zhoudapao@mail.nankai.edu.cn.

Abstract

Permutation tableaux were introduced by Steingrímsson and Williams. Corteel and Kim defined the sign of a permutation tableau in terms of the number of unrestricted columns. The sign-imbalance of permutation tableaux of length n is the sum of signs over permutation tableaux of length n . They have obtained a formula for the sign-imbalance of permutation tableaux of length n by using generating functions and asked for a combinatorial proof. Moreover, they raised the question of finding a sign-imbalance formula for type B permutation tableaux introduced by Lam and Williams. We define a statistic $\overline{\text{wm}}$ over permutations and show that the number of unrestricted columns over permutation tableaux of length n is equally distributed with $\overline{\text{wm}}$ over permutations of length n . This leads to a combinatorial interpretation of the formula of Corteel and Kim. For type B permutation tableaux, we define the sign of a type B permutation tableau in term of the number of certain rows and columns. On the other hand, we construct a bijection between the type B permutation tableaux of length n and symmetric permutations of length $2n$ and we show that the statistic $\overline{\text{wm}}$ over symmetric permutations of length $2n$ is equally distributed with the number of certain rows and columns over type B permutation tableaux of length n . Based on this correspondence and an involution on symmetric permutation of length $2n$, we obtain a sign-imbalance formula for type B permutation tableaux.

Keywords: permutation tableau, sign-imbalance, weak excedance, bijection, signed permutation, symmetric permutation

AMS Subject Classifications: 05A05, 05A15

1 Introduction

This paper is concerned with two questions on the sign-imbalance of permutation tableaux of type A and type B , raised by Corteel and Kim [8]. Permutation tableaux were introduced by Steingrímsson and Williams [17]. They are related to the enumeration of

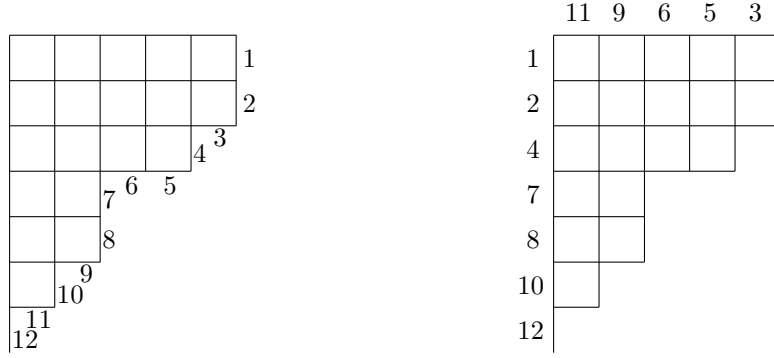


Figure 1.1: The labeling of a Ferrers diagram

totally positive Grassmannian cells [13, 15, 16, 18], as well as a statistical physics model called Partially Asymmetric Exclusion Process (PASEP) [3, 4, 5, 6, 7]. For recent studies of permutation tableaux, see, for example, [1, 2, 8, 10, 14].

A permutation tableau is defined based on the Ferrers diagram of a partition λ for which zero parts are allowed. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition, that is, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$. The Ferrers diagram of λ is a left-justified arrangement with λ_i squares in the i th row. The length of a Ferrers diagram is the total number of rows and columns (including empty rows). In particular, the length of the Ferrers diagram of the empty partition is defined to be zero.

Given a Ferrers diagram F of length n , we label the rows and columns of F as follows. First, we give labels to the steps in the south-east border with $1, 2, \dots, n$ from north-east to south-west. Then we label a row (resp. column) with i if the row (resp. column) contains the south (resp. west) step with label i . Notice that we may place a row label to the left of the first column and place a column label at the top of the first row, see Figure 1.1. A row labeled with i is called row i and a column labeled with j is called column j . We use (i, j) to denote the cell in row i and column j .

For a partition λ , a permutation tableau of shape λ is a 0, 1-filling of the Ferrers diagram of λ satisfying the following conditions:

1. Each column has at least one 1;
2. There is no 0 with a 1 above (in the same column) and a 1 to the left (in the same row).

The length of a permutation tableau is defined to be the length of the corresponding Ferrers diagram. Denote by $\mathcal{PT}(n)$ the set of permutation tableaux of length n . Figure 1.2 illustrates a permutation tableau of length 12.

In their study of combinatorics of permutation tableaux in connection with PASEP, Corteel and Williams [5] introduced the concepts of a row-restricted 0 and an unrestricted row. A 0 in a permutation tableau is said to be row-restricted if there is a 1 above (in the same column). A row is called unrestricted if it does not contain any row-restricted

	11	9	6	5	3
1	0	1	0	0	1
2	0	0	1	0	1
4	1	1	1	1	
7	0	1			
8	0	1			
10	1				
12					

Figure 1.2: A permutation tableau T of length 12

0. Otherwise, it is called a restricted row. Let T be a permutation tableau with k columns, and let $\text{urr}(T)$ be the number of unrestricted rows of T . Besides, Corteel and Williams [5] introduced the weight $\text{wt}(T)$ as the total number of 1's in T minus k , and used the notation $\text{topone}(T)$ for the number of 1's in the first row of T . They defined the polynomial

$$F_{\lambda, \alpha, \beta}(q) = \sum_T q^{\text{wt}(T)} \alpha^{-\text{topone}(T)} \beta^{-\text{urr}(T)+1},$$

where the sum ranges over permutation tableaux T of shape λ . Using the matrix ansatz for the PASEP model, they derived a formula for $F_{\lambda, \alpha, \beta}(q)$. Corteel and Nadeau [10] obtained an explicit formula for the generating function of permutation tableaux of length n with respect to the statistics urr and topone :

$$\sum_{T \in \mathcal{PT}(n)} x^{\text{urr}(T)-1} y^{\text{topone}(T)} = (x+y)_{n-1}, \quad (1.1)$$

where $(x)_n$ denote the rising factorial, that is, $(x)_0 = 1$ and $(x)_n = x(x+1) \cdots (x+n-1)$ for $n \geq 1$.

Corteel and Kim [8] gave two bijective proofs of (1.1). Furthermore, they introduced the concepts of a column-restricted 0 and an unrestricted column. A 0 in a permutation tableau is called column-restricted if there is a 1 to the left (in the same row). In the same vain, one can define unrestricted columns and restricted columns. For a permutation tableau T , let $\text{urc}(T)$ denote the number of unrestricted columns of T . Corteel and Kim obtained the following generating function of permutation tableaux of length n with respect to the statistic urc .

Theorem 1.1 *We have*

$$\sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} t^{\text{urc}(T)} x^n = \frac{1 + E_t(x)}{1 + (t-1)x E_t(x)},$$

where

$$E_t(x) = \sum_{n \geq 1} n(t)_{n-1} x^n.$$

The above generating function leads to a formula for the sign-imbalance of permutation tableaux of length n . For a permutation tableau T , the sign of T is given by $\text{sign}(T) = (-1)^{\text{urc}(T)}$. Let

$$s(n) = \sum_{T \in \mathcal{PT}(n)} \text{sign}(T). \quad (1.2)$$

Setting $t = -1$ in Theorem 1.1, Corteel and Kim derived the following formula for $s(n)$.

Theorem 1.2 *Assume that $n = 4k + r$, where $0 \leq r < 4$. Then*

$$s(n) = \begin{cases} (-1)^k \cdot 2^{2k}, & \text{if } r = 0 \text{ or } r = 1, \\ 0, & \text{if } r = 2, \\ (-1)^{k+1} \cdot 2^{2k+1}, & \text{if } r = 3. \end{cases}$$

Corteel and Kim [8] asked for a combinatorial proof of Theorem 1.2. In answer to this question, we introduce a permutation statistic $\overline{\text{wm}}$. More precisely, for a permutation π , we define $\overline{\text{wm}}(\pi)$ to be the number of element π_i such that $\pi_i < i$ and π_i does not appear in the middle of a decreasing subsequence of length three. We show that the statistic urc over permutation tableaux of length n is equally distributed with the statistic $\overline{\text{wm}}$ over permutations of length n . Moreover, for $n \geq 4$, we build a parity reversing involution on the set of permutations $\pi = \pi_1\pi_2 \cdots \pi_n$ such that $\pi_1\pi_2\pi_3\pi_4 \neq 1342, 1432, 2341, 2431$. Using this involution, we obtain a combinatorial proof of Theorem 1.2.

Moreover, the construction of the aforementioned involution implies the following recurrence relation for $s(n)$.

Theorem 1.3 *For $n \geq 3$,*

$$s(n) = 2s(n-1) - 2s(n-2).$$

Notice that the above recurrence relation along with the initial values $s(1) = 1$ and $s(2) = 0$ also leads to a proof of Theorem 1.2.

The second result of this paper is concerned with a question proposed by Corteel and Kim on permutation tableaux of type B . Type B permutation tableaux were introduced by Lam and Williams [13], and further studied by Corteel and Kim [8], and Corteel, Josuat-vergès and Kim [9]. A type B permutation tableau is defined based on a shifted Ferrers diagram. Let F be a Ferrers diagram with k columns. Unlike the underlying Ferrers diagram of a permutation tableau, for the type B case, both empty rows and empty columns are allowed in a Ferrers diagram. The shifted Ferrers diagram of F , denoted by \bar{F} , is defined to be the diagram obtained from F by adding k rows of size $1, 2, \dots, k$ at the top of the diagram.

For a Ferrers diagram F , the length of the corresponding shifted Ferrers diagram \bar{F} is defined to be the length of F , and the diagonal of \bar{F} is the set of rightmost cells of

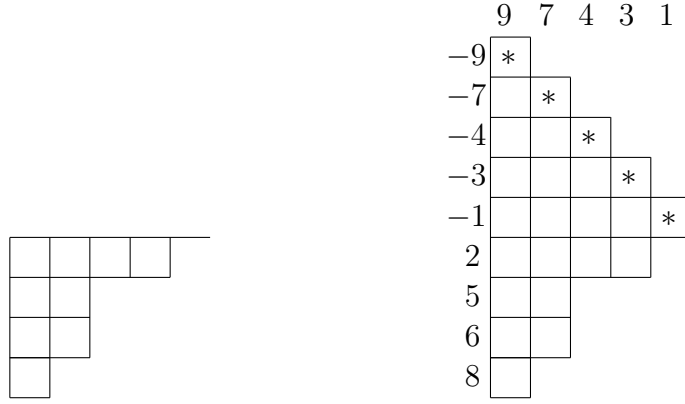


Figure 1.3: A Ferrers diagram and its labeled corresponding shifted Ferrers diagram

the added rows. A diagonal cell is a cell on the diagonal. We label the added row by the opposite number of the label of the column in which the rightmost cell in the row locates. The labels of the other rows and columns remain the same with F . Figure 1.3 illustrates a Ferrers diagram and its corresponding labeled shifted Ferrers diagram, where the diagonal cells are marked with stars.

A type B permutation tableau is a 0, 1-filling of a shifted Ferrers diagram satisfying the following conditions:

1. each column has at least one 1.
2. there is no 0 which has a 1 above (in the same column) and a 1 to the left (in the same row).
3. if a 0 is in a diagonal cell, then it does not have any 1 to the left (in the same row).

The length of a type B permutation tableau is defined to be the length of the corresponding shifted Ferrers diagrams. Let $\mathcal{PT}_B(n)$ be the set of type B permutation tableaux of length n . Figure 1.4 gives a type B permutation tableau of length 8.

Corteel and Kim [8] raised the question of finding a sign-imbalance formula for type B permutation tableaux. In answer to this question, we introduce the concepts of unrestricted rows and columns for a type B permutation tableau. Then we define the sign of a type B permutation tableau by the total number of unrestricted rows and columns. We show that there is a bijection between type B permutation tableaux of length n and symmetric permutations on $\{1, 2, \dots, 2n\}$, where a symmetric permutation π of length $2n$ is a permutation on $[2n]$ such that $\pi_i + \pi_{2n+1-i} = 2n + 1$ for $1 \leq i \leq n$. Using this correspondence, we derive a sign-imbalance formula for type B permutation tableaux of length n .

	8	6	3	2
-8	0			
-6	1	1		
-3	0	0	0	
-2	0	1	0	1
1	1	1	1	1
4	0	1		
5	0	0		
7	1			

Figure 1.4: A type B permutation tableau of length 8.

2 A Combinatorial Proof of Theorem 1.2

In this section, we introduce a permutation statistic $\overline{\text{wm}}$. We show that the statistic $\overline{\text{wm}}$ over permutations of length n is equally distributed with the statistic urc over $\mathcal{PT}(n)$. For $n \geq 4$, we exhibit a parity reversing involution on the set of permutations with the first four elements not equal to 1342, 1432, 2341 nor 2431. Based on this involution, we give a combinatorial proof of Theorem 1.2. Moreover, we derive a recurrence relation for $s(n)$ as given in Theorem 1.3. By taking the initial values of $s(n)$ into consideration, we give another proof of Theorem 1.2.

We first introduce the permutation statistic $\overline{\text{wm}}$. Let $[n] = \{1, 2, \dots, n\}$ and S_n be the set of permutations on $[n]$. Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ of length n , the index i is said to be a weak excedance of π if $\pi_i \geq i$. Otherwise, it is called a non-weak excedance. A permutation π is said to contain a pattern τ if there exists a subsequence of π that has the same relative order as τ . Otherwise, π is said to avoid τ . The element π_i is called a mid-point of π if it is the middle point of a decreasing subsequence of length three of π , namely, there exist $j < i$ and $k > i$ such that $\pi_j > \pi_i > \pi_k$. Otherwise, π_i is called a non-mid-point.

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, we define the following four sets.

$$\begin{aligned}
\text{WM}(\pi) &= \{\pi_i \mid i \text{ is a weak excedance and a mid-point of } \pi\}, \\
\overline{\text{WM}}(\pi) &= \{\pi_i \mid i \text{ is a non-weak excedance and a mid-point of } \pi\}, \\
\text{W}\overline{\text{M}}(\pi) &= \{\pi_i \mid i \text{ is a weak excedance and a non-mid-point of } \pi\}, \\
\overline{\text{W}}\overline{\text{M}}(\pi) &= \{\pi_i \mid i \text{ is a non-weak excedance and a non-mid-point of } \pi\}.
\end{aligned}$$

Set $\overline{\text{wm}}(\pi)$ to be the number of elements in $\overline{\text{WM}}(\pi)$. As an example, given a permutation $\pi = 6, 5, 1, 10, 4, 3, 8, 9, 2, 11, 7, 12$ of length 12, we have $\overline{\text{WM}}(\pi) = \{1, 2, 7\}$ and $\overline{\text{wm}} = 3$.

In the following, we aim to show that the statistic $\overline{\text{wm}}$ over S_n is equally distributed with the statistic urc over $\mathcal{PT}(n)$. To achieve this, we first recall a bijection Φ from $\mathcal{PT}(n)$ to S_n , which was given by Steingrímsson and Williams [17].

A zigzag path on a permutation tableau $T \in \mathcal{PT}(n)$ is a path entering from the left of a row or the top of a column, going to the east or to the south changing the direction alternatively whenever it meets a 1 until exiting the tableau. For convenience, we denote the path entering from i by P_i . Then Φ is defined to be the permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ where $\pi_i = j$ if the zigzag path P_i exits T from row j or column j . As an example, for the permutation tableau T given in Figure 1.2, $\Phi(T) = 6, 5, 1, 10, 4, 3, 8, 9, 2, 11, 7, 12$.

The map Φ has the following properties, of which the proof of the former one can be found in Steingrímsson and Williams [17], and hence, we omit it here.

Proposition 2.1 *Given $T \in \mathcal{PT}(n)$, let $\pi = \Phi(T)$. If i is a weak excedance of π , then i is precisely a row label of T . Otherwise, i is a column label. In particular, π is a permutation with k weak excedance if and only if T has exactly k rows.*

Proposition 2.2 *Let T be a permutation tableau in $\mathcal{PT}(n)$. For $1 \leq i < j \leq n$, if the zigzag paths P_i and P_j intersect, they can only intersect at points, not edges. Moreover, the intersecting points, except the first one, must correspond to a 1 in T .*

Proof. Without loss of generality, we may assume that i is a row label and j is a column label of T . The other cases can be proved similarly. If paths P_i and P_j intersect, let x be the first point they meet. It's easily seen that path P_i travels to x from west, while path P_j travels to x from north. To give a proof of this proposition, we consider two cases.

If x corresponds to a 1 in T , when meeting x , path P_i turns to south and path P_j turns to east. Hence, we see that paths P_i and P_j intersect at the point x , not an edge starting from x . Let y be the second point they meet, if there exists. Clearly, path P_i travels to y from west, while path P_j travels to y from north. We claim that y corresponds to a 1 in T . Otherwise, there will exist a 0 corresponding to y which has a 1 above it and a 1 to its left, which contradicts to the definition of the permutation tableau. Then, by a similar analysis, we may prove that paths P_i and P_j intersect at the point y , not an edge starting from y . Using induction on the number of the intersecting points, we see that in this case paths P_i and P_j can only intersect at points and each intersecting point corresponds to a 1 in T .

If x corresponds to a 0 in T , when meeting x , path P_i goes through x to east, while path P_j goes through x to south. Hence paths P_i and P_j intersect at the point x , not an edge starting from x . Set y to be the second point they meet, if there exists. It is routine to check that P_i travels to y from north, while P_j travels to y from west. By a similar analysis with the above case, we see that P_i and P_j intersect at the point y , which corresponds to a 1 in T . Using induction on the number of the intersecting points, we see that paths P_i and P_j can only intersect at points and each intersecting point, except

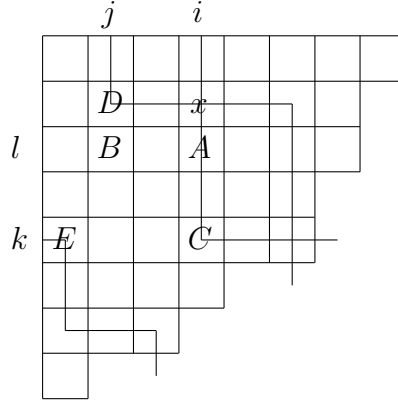


Figure 2.1: The description of P_i , P_j and P_k

the first point, corresponds to a 1 in T . Combining the above two cases, we complete the proof. \blacksquare

Now we are ready to give the proof of the equidistribution of the statistics urc and $\overline{\text{wm}}$. Given a permutation tableau T , let $\text{RR}(T)$, $\text{URR}(T)$, $\text{RC}(T)$, $\text{URC}(T)$ denote the set of the restricted rows, unrestricted rows, restricted columns and unrestricted columns of T , respectively. Clearly, we have $\text{urc}(T) = |\text{URC}(T)|$ and $\text{urr}(T) = |\text{URR}(T)|$. As an example, for the permutation tableau T given in Figure 1.2, we have $\text{URC}(T) = \{3, 9, 11\}$ and $\text{urc}(T) = 3$.

We will show that $\overline{\text{wm}}$ is equally distributed with urc by exhibiting a one-to-one correspondence between $\text{RC}(T)$ and $\overline{\text{WM}}(\pi)$ for a permutation tableau T and $\pi = \Phi(T)$, as given in the following lemma.

Lemma 2.3 *Given a permutation tableau $T \in \mathcal{PT}(n)$, let $\pi = \pi_1\pi_2\ldots\pi_n = \Phi(T)$. There is a bijection between $\text{URC}(T)$ (resp. $\text{RC}(T)$, $\text{RR}(T)$, $\text{URR}(T)$) and $\overline{\text{WM}}(\pi)$ (resp. $\overline{\text{WM}}(\pi)$, $\text{WM}(\pi)$, $\overline{\text{WM}}(\pi)$).*

Proof. By Proposition 2.1, we deduce that $|\text{URC}(T) \cup \text{RC}(T)| = |\overline{\text{WM}}(\pi) \cup \overline{\text{WM}}(\pi)|$. Hence, to prove that there is a bijection between $\text{URC}(T)$ and $\overline{\text{WM}}(\pi)$, it suffices to show that there is a bijection between $\text{RC}(T)$ and $\overline{\text{WM}}(\pi)$. Similarly, to prove that there is a bijection between $\text{URR}(T)$ and $\overline{\text{WM}}(\pi)$, it suffices to prove that there is a bijection between $\text{RR}(T)$ and $\text{WM}(\pi)$. Here, we will provide the bijection between $\text{RC}(T)$ and $\overline{\text{WM}}(\pi)$ only, as the bijection between $\text{RR}(T)$ and $\text{WM}(\pi)$ can be given similarly.

We claim that π is a bijection from $\text{RC}(T)$ to $\overline{\text{WM}}(\pi)$, namely, $i \in \text{RC}(T)$ if and only if $\pi_i \in \overline{\text{WM}}(\pi)$.

On the one hand, we proceed to show that for any $i \in \text{RC}(T)$, we have $\pi_i \in \overline{\text{WM}}(\pi)$. Given $i \in \text{RC}(T)$, it follows from Proposition 2.1 that i is a non-weak excedance. Hence, to prove $\pi_i \in \overline{\text{WM}}(\pi)$, it is left to prove that π_i is a mid-point. Since column i is restricted, there is at least one column-restricted 0 in it. Let A be the lowermost column-restricted 0 in column i , which is located in row l , see Figure 2.1 as a description. Since A is column-restricted, there must be a 1 to the left of A . Denote the nearest 1 to the left of

A by B and suppose B is in column j , where $j > i$. It is easy to see that in column i there is no 1 above A . By the definition of permutation tableau, each column contains at least one 1. So let C be the topmost 1 in column i and assume that C is in row k , where $k < i$. Clearly, we have $k < i < j$, to prove that π_i is a mid-point, we aim to prove that $\pi_k > \pi_i > \pi_j$.

Firstly, we prove that $\pi_j < \pi_i$. We claim that the first intersecting point of the zigzag paths P_i and P_j corresponds to a 0 in T . Let D be the topmost 1 in column j , which is not lower than B . Path P_j travels to D and then turn east. Path P_i travels to C and then turn east. Since B is the nearest 1 to the left of A , the element in cells (l, g) , where $i < g < j$, are all column-restricted 0's. Hence, all the cells above row l and between column j and column i are filled by 0's. This means that the first intersecting point of P_i and P_j , which is denoted by x , corresponds to a 0 in column i . The claim is verified. Moreover, by Proposition 2.2, the following intersecting points, if there exist, must correspond to 1's in T . Then it is not hard to see that after the intersecting point x , path P_j is always on the upper right of P_i . It follows that $\pi_j < \pi_i$, as desired.

Next, we show that $\pi_i < \pi_k$. Assume that E is the leftmost 1 in row k . Clearly, it is not to the right of C . If P_k and P_j do not intersect, then P_i is always on the upper right of P_k . It is obvious that $\pi_i < \pi_k$. Otherwise, suppose the first intersecting point of P_k and P_j is y . We claim that y corresponds to a 1 in T . If point E coincides with C , then clearly P_k and P_j intersect at C , which corresponds to a 1, as desired. If point E is to the left of C , P_k travels to E and then turn south. Then it is not hard to see that path P_i travels to y from north, and path P_k travels to y from west. Hence, y can not be a 0. Otherwise, y will have a 1 above it and a 1 to its left, a contradiction. The claim is verified. Again, by Proposition 2.2, the following intersecting points, if there exist, must correspond to 1's in T . This means that P_i is always on the upper right of P_k . It follows that $\pi_i < \pi_k$, as desired.

On the other hand, we need to show that for each $\pi_i \in \overline{\text{WM}}(\pi)$, we have $i \in \text{RC}(T)$. Since π_i is a mid-point of π , then there exist π_k and π_j such that $k < i < j$ and $\pi_k > \pi_i > \pi_j$. In view of $\pi_i < i$, we deduce that $\pi_j < j$. Let $T = \Phi^{-1}(\pi)$, by Proposition 2.1, i and j are both column labels of T . If there is no column-restricted zero in column i , then the topmost 1 of column i can't be lower than the topmost 1 of column j . By a similar discussion as above, we can obtain $\pi_i < \pi_j$, a contradiction. It follows that $i \in \text{RC}(T)$. This completes the proof. \blacksquare

Example 2.1 Let T be the permutation tableau given by Figure 1.2. Then, $\pi = \Phi(T) = 6, 5, 1, 10, 4, 3, 8, 9, 2, 11, 7, 12$. We have

$$\text{RR}(T) = \{2, 7, 8\} \longleftrightarrow \text{WM}(\pi) = \{5, 8, 9\},$$

$$\text{RC}(T) = \{5, 6\} \longleftrightarrow \overline{\text{WM}}(\pi) = \{3, 4\},$$

$$\text{URR}(T) = \{1, 4, 10, 12\} \longleftrightarrow \overline{\text{WM}}(\pi) = \{6, 10, 11, 12\},$$

$$\text{URC}(T) = \{3, 9, 11\} \longleftrightarrow \overline{\text{WM}}(\pi) = \{1, 2, 7\}.$$

As a consequence of Lemma 2.3, we see that the statistic urc is equidistributed with the statistic $\overline{\text{wm}}$. It follows that

$$s(n) = \sum_{T \in \mathcal{PT}(n)} (-1)^{\text{urc}(T)} = \sum_{\pi \in S_n} (-1)^{\overline{\text{wm}}(\pi)}.$$

To give a proof of Theorem 1.2, we proceed to exhibit a parity reversing involution on a subset of S_n , which are stated in the following lemma.

Lemma 2.4 *There is an involution φ on the subset W_n of S_n , where*

$$W_n = \{\pi \in S_n \mid \pi_1\pi_2\pi_3\pi_4 \neq 1342, 1432, 2341, 2431\}.$$

Moreover, $\overline{\text{wm}}(\pi)$ and $\overline{\text{wm}}(\varphi(\pi))$ have different parities for $\pi \in W_n$.

Before presenting the proof of Lemma 2.4, we first construct an involution on a subset of W_n .

Lemma 2.5 *There is an involution ϕ on the subset V_n of W_n , where*

$$V_n = \{\pi \in S_n \mid \pi_1 \neq 1, 2\} \cup \{\pi \in S_n \mid \pi_1\pi_2 = 12 \text{ or } \pi_1\pi_2 = 21\}.$$

Moreover, $\overline{\text{wm}}(\pi)$ and $\overline{\text{wm}}(\phi(\pi))$ have different parities for $\pi \in V_n$.

Proof. Given a permutation π in V_n , let $\tau = \phi(\pi)$ be given by exchanging the positions of 1 and 2 in π . It is easy to check that ϕ is an involution on V_n . In the following, we proceed to show that ϕ is parity reversing, namely, $\overline{\text{wm}}(\pi)$ and $\overline{\text{wm}}(\tau)$ have different parities.

We may assume that 1 is to the left of 2 in π . For $1 \leq i \leq n$, if $\pi_i \neq 1$ and $\pi_i \neq 2$, it is easy to check that $\pi_i \in \overline{\text{WM}}(\pi)$ if and only if $\tau_i \in \overline{\text{WM}}(\tau)$. Hence, to prove this lemma, we need only to consider the elements 1 and 2 for the two cases below.

- The case $\pi_1 \neq 1, 2$.

Since $\pi_1 \neq 1, 2$ and 1 is to the left of 2 in π , we see that $\pi_2 \neq 2$. This implies that 2 is a non-weak excedance of π . By the fact that 1 is to the left of 2, we see that 2 is a non-mid point of π . It follows that $2 \in \overline{\text{WM}}(\pi)$. Since 2 is to the left of 1 in τ , then $\tau_1 2 1$ forms a 321-pattern of τ . It follows that 2 is a mid-point of τ , which implies that $2 \notin \overline{\text{WM}}(\tau)$. It is not hard to check that $1 \in \overline{\text{WM}}(\pi)$ and $1 \in \overline{\text{WM}}(\tau)$. We conclude that $\overline{\text{wm}}(\tau) = \overline{\text{wm}}(\pi) - 1$. This completes the proof of this case.

- The case $\pi_1\pi_2 = 12$.

It is easily checked that $1, 2 \notin \overline{\text{WM}}(\pi)$, $2 \notin \overline{\text{WM}}(\tau)$, while $1 \in \overline{\text{WM}}(\tau)$. This means that $\overline{\text{wm}}(\tau) = \overline{\text{wm}}(\pi) + 1$. Hence, $\overline{\text{wm}}(\pi)$ and $\overline{\text{wm}}(\tau)$ have different parities.

Combing the above two cases, we complete the proof. \blacksquare

Now, we proceed to give the proof of Lemma 2.4.

Proof of Lemma 2.4. By Lemma 2.5, we have given an involution ϕ on V_n . Hence, to prove this lemma, it is left to construct a parity reversing involution θ on U_n , where $U_n = W_n/V_n$. Clearly, we have $U_n = \{\pi \in W_n \mid \pi_1 = 1 \text{ and } \pi_2 \neq 2\} \cup \{\pi \in W_n \mid \pi_1 = 2 \text{ and } \pi_2 \neq 1\}$. Without loss of generality, we assume that $\pi_1 = 1$. The proof of the cases that $\pi_1 = 2$ can be performed similarly.

Given a permutation π in U_n with $\pi_1 = 1$, we proceed to construct its image under the map θ . If $\pi_1\pi_2\pi_3\pi_4 = 1324$, let $\tau = \theta(\pi)$, where τ is obtained from π by exchanging the positions of 3 and 4. It is easily checked that in this case we have $\theta^2(\pi) = \pi$ and $\overline{\text{wm}}(\pi) = \overline{\text{wm}}(\tau) - 1$.

If $\pi_1\pi_2\pi_3\pi_4 \neq 1324$ and $\pi_1\pi_2\pi_3\pi_4 \neq 1423$, let $\pi_i\pi_j\pi_k$ be the subsequence of π containing 2, 3 and 4. Set $\tau = \theta(\pi)$, where τ is obtained from π by exchanging the positions of π_j and π_k . Clearly, in this case we have $\theta^2(\pi) = \pi$. In the following, we aim to compute the relations between $\overline{\text{wm}}(\pi)$ and $\overline{\text{wm}}(\tau)$. To achieve this, we consider two cases.

- The case $j > 3$.

We may assume that $\pi_j < \pi_k$. By the definition of θ , we see that $\tau_i\tau_j\tau_k = \pi_i\pi_k\pi_j$. By a routine analysis, we can obtain that $\pi_j \in \overline{\text{WM}}(\pi)$, $\pi_k \in \overline{\text{WM}}(\pi)$ and $\pi_j \in \overline{\text{WM}}(\tau)$. If $i > 2$, then $\tau_2\pi_k\pi_j$ forms a 321 pattern of τ . If $i = 2$, then $\tau_3\pi_k\pi_j$ forms a 321 pattern of τ . Thus, we conclude that $\pi_k \notin \overline{\text{WM}}(\tau)$. It follows that in this case we have $\overline{\text{wm}}(\pi) = \overline{\text{wm}}(\tau) + 1$.

- The case $j = 3$.

By the definition of U_n and the assumption that $\pi_1\pi_2\pi_3\pi_4 \neq 1324, 1423$, we deduce that $i = 2$ and $k > 4$. We may assume that $\pi_j < \pi_k$. Since $\pi_2 \neq 2$, we have $\pi_j = 2$. Then it is routine to check that $\pi_j \in \overline{\text{WM}}(\pi)$, $\pi_k \in \overline{\text{WM}}(\pi)$ and $\pi_j \in \overline{\text{WM}}(\tau)$. By the fact that $\pi_k \geq 3$, we see that $\tau_3 = \pi_k$ is a weak excedance of τ . Hence, $\pi_k \notin \overline{\text{wm}}(\tau)$. It follows that in this case we have $\overline{\text{wm}}(\pi) = \overline{\text{wm}}(\tau) + 1$.

Combing the above two cases, we complete the proof. \blacksquare

Based on Lemma 2.3 and Lemma 2.4, we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. Set $n = 4k + r$ ($0 \leq r < 4$). Following from Lemma 2.3, it suffices to prove that

$$s(n) = \sum_{\pi \in S_n} (-1)^{\overline{\text{wm}}(\pi)} = \begin{cases} (-1)^k \cdot 2^{2k}, & \text{if } r = 0 \text{ or } r = 1, \\ 0, & \text{if } r = 2, \\ (-1)^{k+1} \cdot 2^{2k+1}, & \text{if } r = 3. \end{cases}$$

Let R_n be the set of permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ such that $\pi_{4i-3}\pi_{4i-2}\pi_{4i-1}\pi_{4i}$ is a permutation of $\{4i-3, 4i-2, 4i-1, 4i\}$ that is order isomorphic to 1342, 1432, 2341

or 2431 for $1 \leq i \leq k$. In the following, we proceed to construct a parity reversing involution χ on S_n/R_n .

Assume that π is a permutation in S_n/R_n . By the definition of R_n , we see that there exist integers $i(1 \leq i \leq k)$ such that $\pi_{4i-3}\pi_{4i-2}\pi_{4i-1}\pi_{4i}$ is not a permutation of $\{4i-3, 4i-2, 4i-1, 4i\}$ that is order isomorphic to 1342, 1432, 2341 or 2431. Let j be the minimum element among these integers. Suppose that p is the permutation obtained from π by deleting the first $4j-4$ elements and subtracting each remaining element by $4j-4$. Clearly, p is a permutation in W_m , where $m = 4(k-j) + 4 + r$. Write $q = \theta(p)$. We define χ by letting $\tau = \tau_1\tau_2 \cdots \tau_n = \chi(\pi)$, where $\tau_1\tau_2 \cdots \tau_{4j-4} = \pi_1\pi_2 \cdots \pi_{4j-4}$ and $\tau_i = q_{i-4j+4} + 4j-4$ ($i \geq 4j-4$). By the fact that θ is a parity reversing involution, it is routine to check that χ is a parity reversing involution. It follows that

$$\sum_{\pi \in S_n/R_n} (-1)^{\overline{\text{wm}}(\pi)} = 0.$$

Thus, we have $s(n) = \sum_{\pi \in R_n} (-1)^{\overline{\text{wm}}(\pi)}$. To compute $s(n)$, we consider the following four cases.

- The case $r = 0$, namely, $n = 4k$

Assume that π is a permutation in R_{4k} . For $1 \leq i \leq k$, we have $\{\pi_i, \pi_{i+1}, \pi_{i+2}, \pi_{i+3}\}$ is order isomorphic to 1342, 1432, 2341 or 2431. Then, it is not hard to check that there exists exactly one element in $\{\pi_i, \pi_{i+1}, \pi_{i+2}, \pi_{i+3}\}$ which is contained in the set $\overline{\text{WM}}(\pi)$. It follows that $\overline{\text{wm}}(\pi) = (-1)^k$. Since $|R_{4k}| = 2^{2k}$, we have $s(n) = (-1)^k \cdot 2^{2k}$ for $n = 4k$.

- The case $r = 1$, namely, $n = 4k + 1$

Given a permutation π in R_{4k+1} , by the definition of R_{4k+1} , we see that $\pi_n = n$. Clearly, $n \notin \overline{\text{WM}}(\pi)$. We deduce that $s(n) = (-1)^k \cdot 2^{2k}$ for $n = 4k + 1$.

- The case $r = 2$, namely, $n = 4k + 2$

Given a permutation π in R_{4k+2} , by the definition of R_{4k+2} , we see that $\pi_{n-1}\pi_n$ is a permutation of the set $\{n-1, n\}$. Assume that $\pi_{n-1} = n-1$ and $\pi_n = n$. Setting τ to be the permutation obtained from π by exchanging the positions of $n-1$ and n . It is easily seen that $\overline{\text{wm}}(\pi) = \overline{\text{wm}}(\tau) - 1$. It follows that $s(n) = 0$ for $n = 4k + 2$.

- The case $r = 3$, namely, $n = 4k + 3$

Let π be the permutation in R_{4k+3} . By the definition of R_{4k+3} , we see that $\pi_{n-2}\pi_{n-1}\pi_n$ is a permutation of $\{n-2, n-1, n\}$. If $\pi_{n-2}\pi_{n-1}\pi_n$ is order isomorphic to 123 or 321, let τ be the permutation obtained from π by exchanging the positions of $n-2$ and $n-1$. It is routine to check that $\overline{\text{wm}}(\pi) = \overline{\text{wm}}(\tau) - 1$. Thus, $s(n)$ equals to the sign-imbalance of the set R'_{4k+3} , which is a subset of R_{4k+3} with $\pi_{n-2}\pi_{n-1}\pi_n$ order isomorphic to 132 or 231. Notice that $\overline{\text{wm}}(\pi) = (-1)^{k+1}$ for each $\pi \in R'_{4k+3}$ and $|R'_{4k+3}| = 2^{2k+1}$. We conclude that $s(n) = (-1)^{k+1} \cdot 2^{2k+1}$ for $n = 4k + 3$.

Combining the above four cases, we complete the proof. \blacksquare

Up to now, we have given a combinatorial proof of Theorem 1.2. In fact, there is another partially combinatorial proof of this theorem. We first derive the following recurrence relation for $s(n)$ as given in Theorem 1.3.

Proof of Theorem 1.3. By Lemma 2.5, we deduce that $\sum_{\pi \in V_n} (-1)^{\overline{\text{wm}}(\pi)} = 0$. Hence, to obtain a recurrence relation for $s(n)$, it is left to deal with the remaining cases for $\pi \in S_n/V_n$. Clearly, $S_n/V_n = \{\pi \in S_n \mid \pi_1 = 1 \text{ and } \pi_2 \neq 2\} \cup \{\pi \in S_n \mid \pi_1 = 2 \text{ and } \pi_2 \neq 1\}$.

Given a permutation π with $\pi_1 = 1$ and $\pi_2 \neq 2$, let $\tau = \tau_1\tau_2 \cdots \tau_{n-1}$ be the permutation given by $\tau_i = \pi_{i+1} - 1$. Since 1 is a weak excedance of π , we see that $1 \notin \overline{\text{WM}}(\pi)$. It follows that $\overline{\text{wm}}(\tau) = \overline{\text{wm}}(\pi)$. Notice that $\tau_1 \neq 1$, we deduce that

$$\begin{aligned} \sum_{\substack{\pi \in S_n \\ \pi_1=1 \text{ and } \pi_2 \neq 2}} (-1)^{\overline{\text{wm}}(\pi)} &= \sum_{\tau \in S_{n-1}, \tau_1 \neq 1} (-1)^{\overline{\text{wm}}(\tau)} \\ &= s(n-1) - s(n-2). \end{aligned}$$

Moreover, it is routine to check that the sign-imbalance of the set $\{\pi \in S_n \mid \pi_1 = 1 \text{ and } \pi_2 \neq 2\}$ is equal to that of the set $\{\pi \in S_n \mid \pi_1 = 2 \text{ and } \pi_2 \neq 1\}$. Hence, we conclude that $s(n) = 2s(n-1) - 2s(n-2)$. This completes the proof. \blacksquare

Notice that $s(1) = 1$ and $s(2) = 0$. By Theorem 1.3, we give another proof of Theorem 1.2.

3 The sign-imbalance of permutation tableaux of type B

In this section, we define the sign of a type B permutation tableau T , which is denoted by $\text{sign}_B(T)$. We show that there is a bijection between type B permutation tableaux of length n and symmetric permutations on $[2n]$. Using this correspondence, we derive a sign-imbalance formula for type B permutation tableaux, which is given as follows.

Theorem 3.1 *If $n = 2k + r$ ($0 \leq r < 2$), then*

$$s_B(n) = \sum_{T \in \mathcal{PT}_B(n)} \text{sign}_B(T) = \begin{cases} 2^{\frac{n}{2}}, & \text{if } r = 0, \\ 0, & \text{if } r = 1. \end{cases}$$

We first introduce two symmetric constructions corresponding to a type B permutation tableau T of length n . Add the cells obtained by reflecting the non-diagonal cells in T about the diagonal line to T , we get a symmetric construction T_s , which we call symmetric tableau. It is easily seen that the length of the symmetric tableau T_s is $2n$. Label

the steps in the south-east border of T_s with $1, 2, \dots, 2n$ from north-east to south-west. We label a row (resp. column) of T_s with i if the row (resp. column) contains the south (resp. west) step with label i . By condition 3 in the definition of type B permutation tableaux, we see that there is no 0 in T_s which has a 1 above (in the same column) and a 1 to the left (in the same row). Hence, if we remove all the rows and columns of T_s that have no 1's and keep the labels unchanged, we obtain a symmetric permutation tableau of type A , which is denoted T_A . Here, we remark that in this paper we always use T_s and T_A to present the corresponding symmetric tableau and symmetric permutation tableau of T , respectively. As an example, for the type B permutation tableau T in Figure 1.4, its corresponding T_s and T_A are given in Figure 3.1.

		16	14	11	10	8	5	4	2
1	0	1	0	0	1	0	0	1	
3	1	1	0	1	1	1	0		
6	0	0	0	0	1				
7	0	1	0	1	1				
9	1	1	1	1					
12	0	1							
13	0	0							
15	1								

		16	14	11	10	8	5	2
1	0	1	0	0	1	0	1	
3	1	1	0	1	1	1		
6	0	0	0	0	1			
7	0	1	0	1	1			
9	1	1	1	1				
12	0	1						
15	1							

Figure 3.1: The corresponding symmetric tableau T_s (left) and symmetric permutation tableau T_A (right) of T given in Figure 1.4.

Given a type B permutation tableau T , we say a 0 in T is row-restricted if it has a 1 above (in the same column). While, a 0 is said to be column-restricted if it has a 1 to the left (in the same row). Note that in this paper a 0 in the diagonal cell is not treated as row-restricted, which is different from the definitions given by Corteel and Kim [9]. We define the set of type B unrestricted rows $\text{URR}_B(T)$ and the set of type B unrestricted columns $\text{URC}_B(T)$ of T as follows.

$\text{URR}_B(T) = \{i \mid i \text{ is a positive row label satisfies that row } i \text{ contains no row-restricted 0 and contains at least one 1}\},$

$\text{URC}_B(T) = \{j \mid j \text{ is a column label satisfies that column } j \text{ contains no column-restricted 0 and row } -j \text{ contains no row-restricted 0}\}.$

Let $\text{UR}_B(T) = \text{URR}_B(T) \cup \text{URC}_B(T)$ and $\text{ur}_B(T) = |\text{UR}_B(T)|$. Define the sign of the type B permutation tableau T by

$$\text{sign}_B(T) = (-1)^{\text{ur}_B(T)}.$$

By the construction of T_A from T , it is easy to check that $\text{ur}_B(T) = \text{urc}(T_A)$. It follows that

$$\text{sign}_B(T) = (-1)^{\text{urc}(T_A)}.$$

Let SP_{2n} be the set of the symmetric permutations on $[2n]$. In the following, we shall construct a bijection Φ_B between $\mathcal{PT}_B(n)$ and SP_{2n} , which allows us to translate the statistic ur_B on $\mathcal{PT}_B(n)$ to the statistic \overline{wm} on SP_{2n} .

Let T be a type B permutation tableau of length n . We define $\pi = \Phi_B(T)$ to be the permutation on $[2n]$ which is obtained from T_s by using the zigzag map. Let S be the set of labels of the rows and columns in T_s which contain no 1. In fact, $\pi = \Phi_B(T)$ can be also obtained by computing $\pi' = \Phi(T_A)$ on $[2n] \setminus S$ and then setting the element in S to be fixed points of π . As an example, for the type B permutation tableau T given in Figure 1.4, we have that $\pi = \Phi_B(T) = 8, 1, 12, 4, 3, 7, 11, 2, 15, 6, 10, 14, 13, 5, 16, 9$. While $S = \{4, 13\}$ and

$$\pi' = \begin{pmatrix} 1 & 2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 14 & 15 & 16 \\ 8 & 1 & 12 & 3 & 7 & 11 & 2 & 15 & 6 & 10 & 14 & 5 & 16 & 9 \end{pmatrix}.$$

It should be remarked that $\pi'(i)$ is said to be a weak excedance if $\pi'(i) \geq i$. Otherwise, $\pi'(i)$ is said to be a non-weak excedance.

To show that Φ_B is a bijection from $\mathcal{PT}_B(n)$ to SP_{2n} , we need the following proposition.

Proposition 3.2 *Let $T \in \mathcal{PT}_B(n)$ and $\pi = \Phi_B(T)$. If i is a fixed point of π , then i is a column label of T_s when $1 \leq i \leq n$ and i is a row label of T_s when $n+1 \leq i \leq 2n$. If π_i is an excedance, then i is a row label of T_s , and if π_i is a non-weak excedance, then i is a column label of T_s .*

Proof. Since $T \in \mathcal{PT}_B(n)$, we see that there is at least one 1 in each column of T . It follows that each row in T_s with label $1 \leq i \leq n$ and each column in T_s with label $n+1 \leq j \leq 2n$ contain at least one 1. Hence if i is a fixed point of $\pi = \Phi_B(T)$, then i is a column label of T_s when $1 \leq i \leq n$ and i is a row label of T_s when $n+1 \leq i \leq 2n$. Since the non-fixed points of π is the permutation $\Phi(T_A)$, then the remaining parts of the proposition follows directly from Proposition 2.1. The proof is completed. ■

Based on Proposition 3.2, we have the following theorem.

Theorem 3.3 *Φ_B is a bijection from $\mathcal{PT}_B(n)$ to SP_{2n} .*

Proof. Given a type B permutation tableau $T \in \mathcal{PT}_B(n)$, let $\pi = \Phi_B(T)$. We claim that π is a symmetric permutation on $[2n]$. Since T_s is a symmetric tableau, the zigzag paths P_i and P_{2n+1-i} is symmetric about the diagonal line of T_s . Then, it is easy to check that $\pi_i + \pi_{2n+1-i} = 2n+1$ for $1 \leq i \leq 2n$. Hence, π is a symmetric permutation on $[2n]$, as claimed.

To prove that Φ_B is a bijection, we will give an explicit description of its inverse. Given $\pi \in SP_{2n}$, set T to be the type B permutation tableau obtained by the following procedure.

First, we construct the symmetric tableau T_s corresponding to T . Compute the fixed points, excedances and non-weak excedances of π . Using Proposition 3.2, it is easy to obtain a Ferrers diagram F , which is the shape of T_s . Hence, to get T_s , we need to fill the cells of F with 0's or 1's. If i is a fixed point of π , fill the cells in row i or column i with 0's. Denote F_A the Ferrers diagram obtained by removing all the rows and columns which have been already filled. It should be noted that the labels of the columns and rows in F_A remain the same with that of F . Let π' be the permutation obtained from π by deleting all the fixed points. Let $T_A = \Phi^{-1}(\pi')$. It is not hard to check that F_A is the shape of T_A . Then, fill F_A with 0's and 1's such that it equals to T_A . Thus, we have constructed T_s . And the type B permutation tableau T can be obtained by removing all the cells on the upper right of the diagonal line of T_s . It is clear from the above construction that T is a type B permutation tableau, and moreover, it is the inverse image of $\Phi_B^{-1}(\pi)$. This completes the proof. ■

Now, we are ready to translate the statistic ur_B to the statistic $\overline{\text{wm}}$.

Lemma 3.4 *The statistic ur_B on $\mathcal{PT}_B(n)$ is equaidistributed with the statistic $\overline{\text{wm}}$ on SP_{2n} .*

Proof. Given $T \in \mathcal{PT}_B(n)$, we write $\pi = \Phi_B(T)$. Recall that $\text{ur}_B(T) = |\text{URC}(T_A)|$. Hence, to prove this lemma, we need only to show that there is a bijection between $\text{URC}(T_A)$ and $\overline{\text{WM}}(\pi)$. By the definition of the map Φ_B , we see that π can be obtained from $\pi' = \Phi(T_A)$ by setting all the other elements in $[2n]$ as fixed points. Following from Lemma 2.3, there is a bijection between $\text{URC}(T_A)$ and $\overline{\text{WM}}(\pi')$. Thus, to prove this lemma, it suffices to show that $\overline{\text{WM}}(\pi') = \overline{\text{WM}}(\pi)$.

Firstly, we claim that if $\pi_i \in \overline{\text{WM}}(\pi')$, then $\pi_i \in \overline{\text{WM}}(\pi)$. Since π_i is a non-weak excedance of π' , there exists $j < i$ such that $\pi_j > \pi_i$. By the fact that π_i is a non-mid point of π' , we deduce that $\pi_l > \pi_i$ for any non-fixed point $l > i$ of π . For fixed point $k > i$ of π , it is easy to see that $\pi_k > \pi_i$. Hence, all the elements that are after π_i in π are larger than π_i . It follows that π_i is a non-mid point of π . Since π_i is a non-weak excedance of π' , then π_i is a non-weak excedance of π . Thus, $\pi_i \in \overline{\text{WM}}(\pi)$. The claim is verified.

Notice that the fixed points of π are weak excedance of π . It is easy to verify that if $\pi_i \in \overline{\text{WM}}(\pi)$, then $\pi_i \in \overline{\text{WM}}(\pi')$. Hence, we conclude that $\overline{\text{WM}}(\pi') = \overline{\text{WM}}(\pi)$. This completes the proof. ■

Based on Lemma 3.4, we proceed to give a proof of Theorem 3.1. Notice that the proof of this theorem is similar to that of Theorem 1.2. We just outline the main idea of the proof, details are omitted.

Proof of Theorem 3.1. By Lemma 3.4, we see that Theorem 3.1 is equivalent to

$$s_B(n) = \sum_{\pi \in SP_{2n}} (-1)^{\overline{\text{wm}}(\pi)} = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (3.1)$$

Define a subset of SP_{2n} as follows

$$SS_{2n} = \{\pi \in SP_{2n} | \pi_1\pi_2 \neq 12 \text{ and } \pi_1\pi_2 \neq 21\}.$$

For the case that $\pi \in SS_{2n}$, exchanging the positions of 1, 2 and the positions of $2n-1, 2n$ to get τ . It can be shown that the parities of $\overline{\text{wm}}(\pi)$ and $\overline{\text{wm}}(\tau)$ are different. The proof of this fact can be performed similarly to that of Lemma 2.5. It follows that

$$\sum_{\pi \in SS_{2n}} (-1)^{\overline{\text{wm}}(\pi)} = 0.$$

Hence, we deduce that

$$\begin{aligned} s_B(n) &= \sum_{\pi \in SP_{2n}/SS_{2n}} (-1)^{\overline{\text{wm}}(\pi)} \\ &= \sum_{\substack{\pi \in SP_{2n} \\ \pi_1=1 \text{ and } \pi_2=2}} (-1)^{\overline{\text{wm}}(\pi)} + \sum_{\substack{\pi \in SP_{2n} \\ \pi_1=2 \text{ and } \pi_2=1}} (-1)^{\overline{\text{wm}}(\pi)} \\ &= \sum_{\pi' \in SP_{2n-4}} (-1)^{\overline{\text{wm}}(\pi')} + \sum_{\pi'' \in SP_{2n-4}} (-1)^{\overline{\text{wm}}(\pi'')} \\ &= 2s_B(n-2). \end{aligned}$$

Taking the initial values into consideration, we arrive at (3.1). This completes the proof. ■

References

- [1] A. Burstein, On some properties of permutation tableaux, *Ann. Combin.* 11 (2007), 355–368.
- [2] W.Y.C. Chen and L.H. Liu, Permutation tableaux and the dashed permutation pattern 32-1, *Electron. J. Combin.* 18 (2011), 111–122.
- [3] S. Corteel, Crossings and alignments of permutations, *Adv. Appl. Math.* 32 (2007), no 2, 149–163.
- [4] S. Corteel, R. Brak, A. Rechnitzer and J. Essam, A combinatorial derivation of the PASEP stationary state, *Electron. J. Combin.* 13 (2006), R108.
- [5] S. Corteel and L. Williams, Tableaux combinatorics for the asymmetric exclusion process I, *Adv. Appl. Math.* 37 (2007), 293–310.
- [6] S. Corteel and L. Williams, A Markov chain on permutations which projects to the PASEP, *Int. Math. Res. Not.* (2007), Art. ID rnm055.

- [7] S. Corteel and L. Williams, Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials, *Duke Math J.* 159 (2011), 385–415.
- [8] S. Corteel and J.S. Kim, Combinatorics on Permutation tableaux of type A and type B, *European J. Combin.* 32 (2011), 563–579.
- [9] S. Corteel, M. Josuat-vergès and J.S. Kim, Combinatorics of the permutation tableaux of type B, [arXiv:1203.0154](#).
- [10] S. Corteel and P. Nadeau, Bijections for permutation tableaux, *European J. Combin.* 30 (2009), 295–300.
- [11] M.-P. Delest and G. Viennot, Algebraic languages and polynomials enumeration, *Theoret. Comput. Sci.* 34 (1984), 169–206.
- [12] E.S. Egge, Restricted symmetric permutations, *Ann. Combin.* 11 (2007), 405–434.
- [13] T. Lam and L. Williams, Total positivity for cominuscule Grassmannians, *New York J. Math.* 14 (2008), 53–99.
- [14] P. Nadeau, the structure of alternative tableaux, *J. Combin. Theory Ser. A* 114 (2007), 211–234.
- [15] A. Postnikov, Total positivity, Grassmannians, and networks, [arXiv:math/0609764](#).
- [16] J.S. Scott, Grassmannians and cluster algebras, *Proc. London Math. Soc.* 92 (2006), 345–380.
- [17] E. Steingrímsson and L. Williams, Permutation tableaux and permutation patterns, *J. Combin. Theory Ser. A* 114 (2007), 211–234.
- [18] L. Williams, Enumeration of totally positive Grassmann cells, *Adv. Math.* 190 (2005), 319–342.